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# Pseudo-Kähler metrics on six-dimensional nilpotent Lie algebras

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**Abstract**

It is well known that any symplectic manifold $(M, \Omega)$ has an almost complex structure $J$ which is compatible with $\Omega$. In this paper, we deal with the existence of compatible pairs $(J, \Omega)$ on nilpotent Lie algebras $\mathfrak{g}$ of dimension $\leq 6$, $J$ being an *integrable* almost complex structure. We prove that if such a pair exists, $J$ must satisfy some extra conditions, namely $J$ must be nilpotent in the sense of [Trans. Am. Math. Soc. 352 (2000) 5405]. Associated to any such a compatible pair, there is a pseudo-Kähler metric $g$ which cannot be positive definite unless $\mathfrak{g}$ be abelian. All these metrics are Ricci flat, although many of them are nonflat, and we study the behaviour of its curvature tensor under deformation.


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## 1. Introduction

A *pseudo-Kähler* (or *indefinite Kähler*) *structure* $(J, \Omega)$ on a Lie algebra $\mathfrak{g}$ consists of a nondegenerate closed 2-form $\Omega$ and a complex structure $J$ on $\mathfrak{g}$ which are *compatible*, i.e.

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$\Omega(JX, JY) = \Omega(X, Y)$, for all $X, Y \in \mathfrak{g}$. Here, by a complex structure $J$ on $\mathfrak{g}$ we mean an endomorphism of $\mathfrak{g}$ such that $J^2 = -\mathrm{Id}$, and without torsion in the sense that the Nijenhuis tensor of $J$ vanishes. Given a pseudo-Kähler structure $(J, \Omega)$ on $\mathfrak{g}$, there exists an associated nondegenerate symmetric 2-tensor $g$ on $\mathfrak{g}$ defined by $g(X, Y) = \Omega(X, JY)$ for $X, Y \in \mathfrak{g}$. We shall say that $g$ is a *pseudo-Kähler metric* on the Lie algebra $\mathfrak{g}$.

It is well known [1,7] that if the Lie algebra $\mathfrak{g}$ is nilpotent then the metric $g$ associated to any compatible pair $(J, \Omega)$ cannot be positive definite, unless $\mathfrak{g}$ be abelian. However, examples of nilpotent Lie algebras with pseudo-Kähler metrics abound in the literature, although as far as we know a general classification result is not available if $\dim \mathfrak{g} \geq 6$. In dimension 4 it is already known that, apart from the abelian Lie algebra, only the Lie algebra $\mathfrak{Kt}$ underlying the Kodaira–Thurston nilmanifold [9] possesses compatible pairs $(J, \Omega)$; actually, any complex structure $J$ on $\mathfrak{Kt}$ admits a compatible symplectic form $\Omega$.

Our goal in this paper is to classify six-dimensional nilpotent Lie algebras $\mathfrak{g}$ admitting pseudo-Kähler metrics. Since these metrics are in one-to-one correspondence with compatible pairs $(J, \Omega)$ on $\mathfrak{g}$, we introduce the following spaces. For any complex structure $J$ fixed on $\mathfrak{g}$, we denote by $\mathcal{S}_c(\mathfrak{g}, J)$ the set of all symplectic forms $\Omega$ on $\mathfrak{g}$ which are compatible with $J$. Each set $\mathcal{S}_c(\mathfrak{g}, J)$ is open in the vector space consisting of all real closed 2-forms of bidegree $(1, 1)$ with respect to $J$, and therefore, if it is nonempty, its dimension can be computed explicitly. Hence, the existence of a pseudo-Kähler metric on $\mathfrak{g}$ is equivalent to prove that $\mathcal{S}_c(\mathfrak{g}, J) \neq \emptyset$ for some $J \in \mathcal{C}(\mathfrak{g})$, where $\mathcal{C}(\mathfrak{g})$ consists of all complex structures on the Lie algebra $\mathfrak{g}$.

The paper is structured as follows. In Section 2, starting from Salamon's characterization [8] of the existence of complex structures on six-dimensional nilpotent Lie algebras $\mathfrak{g}$, we prove in Theorem 2.5 that the complex structure $J$ underlying any pseudo-Kähler metric on $\mathfrak{g}$ must be *nilpotent* in the sense of [3]. Therefore, $\mathcal{S}_c(\mathfrak{g}, J)$ is empty for any nonnilpotent $J$ on $\mathfrak{g}$. Using this fact, Corollary 2.8 exhibits a nilpotent Lie algebra having complex structures $J$ and symplectic forms $\Omega$, but admitting no compatible pair $(J, \Omega)$. To our knowledge, this is the first known example of such a situation on a nilpotent Lie algebra.

Section 3 is devoted to classify six-dimensional nilpotent Lie algebras with complex structures $J$ admitting compatible symplectic forms $\Omega$. The algebras are listed in Theorem 3.1, and along its proof we construct explicit pseudo-Kähler metrics when they exist. Our classification result can be summarized as follows: in dimension 6, if $\mathfrak{g}$ has symplectic forms and nilpotent complex structures, then there always exists a compatible pair $(J, \Omega)$ on $\mathfrak{g}$.

Since any pseudo-Kähler metric on $\mathfrak{g}$ is Ricci flat [6], Propositions 3.5–3.10 provide many explicit examples of Ricci flat pseudo-Riemannian metrics and, even more, many among these metrics are nonflat as it is shown in Theorem 4.1. We also study in Section 4.1 the variation of the curvature tensor along curves consisting of pseudo-Kähler metrics, showing how a flat metric can be deformed to a nonflat one, and vice versa, on a particular Lie algebra.

Finally, notice that if $G$ is a simply connected nilpotent Lie group admitting a lattice $\Gamma$ of maximal rank, then any left invariant pseudo-Kähler metric on $G$ induces a pseudo-Kähler metric on the compact nilmanifold $\Gamma\backslash G$ in a natural way. In fact, $\Omega$ descends to a symplectic form on $\Gamma\backslash G$ which is compatible with the complex structure induced on this quotient. Since left invariant pseudo-Kähler metrics on $G$ are canonically identified with pseudo-Kähler metrics on its Lie algebra $\mathfrak{g}$, it suffices to work at the Lie algebra level in order to investigate some properties of the corresponding metric on the compact nilmanifold $\Gamma\backslash G$ (see Section 4.2 for more details).

## 2. On the nilpotency of the complex structure

Let $\mathfrak{g}$ be a Lie algebra, and denote by $[\cdot,\cdot]$ its bracket. The *descending central series* $\{\mathfrak{g}^k\}_{k\geq 0}$ of $\mathfrak{g}$ is defined inductively by

$$\mathfrak{g}^0=\mathfrak{g},\qquad \mathfrak{g}^k=[\mathfrak{g}^{k-1},\mathfrak{g}],\quad k\geq 1.$$

The Lie algebra $\mathfrak{g}$ is said to be *nilpotent* if $\mathfrak{g}^k=0$ for some $k$.

A *complex structure* $J$ on a nilpotent Lie algebra $\mathfrak{g}$ is an endomorphism $J:\mathfrak{g}\to\mathfrak{g}$ of the Lie algebra such that $J^2=-\mathrm{Id}$, and

$$[JX,JY]=J[JX,Y]+J[X,JY]+[X,Y]$$

for any $X,Y\in\mathfrak{g}$. Associated to $J$, there exists an ascending series $\{\mathfrak{a}_l(J)\}_{l\geq 0}$ on the Lie algebra, defined inductively by

$$\mathfrak{a}_0(J)=\{0\},\qquad \mathfrak{a}_l(J)=\{X\in\mathfrak{g}\,|\,[X,\mathfrak{g}]\subseteq\mathfrak{a}_{l-1}(J)\text{ and }[JX,\mathfrak{g}]\subseteq\mathfrak{a}_{l-1}(J)\},\quad l\geq 1.$$

If $\mathfrak{a}_l(J)=\mathfrak{g}$ for some $l$, then the complex structure $J$ is called *nilpotent* [3].

There are two special classes of nilpotent complex structures. A complex structure $J$ satisfying $[JX,JY]=[X,Y]$ for all $X,Y\in\mathfrak{g}$, is obviously nilpotent and it is called *abelian*.

On the other hand, if $\mathfrak{g}$ is complex as a Lie algebra then its canonical complex structure $J$ satisfies $[JX,Y]=J[X,Y]$ for all $X,Y\in\mathfrak{g}$ and it is clearly nilpotent. However, in [4] it is proved that any closed 2-form compatible with such a $J$ is always degenerate.

In addition to describing the Lie algebra $\mathfrak{g}$ in terms of its bracket $[\cdot,\cdot]$, we shall mostly use the Chevalley–Eilenberg differential d on the dual $\mathfrak{g}^*$. Since $d\omega(X,Y)=-\omega([X,Y])$, the two descriptions are equivalent.

When a complex structure $J$ is fixed on $\mathfrak{g}$, there is a natural bigraduation induced on the spaces $\wedge^k_{\mathbb{C}}(\mathfrak{g}^*)=\oplus_{p+q=k}\wedge^{p,q}(\mathfrak{g}^*)$, where $\wedge^k_{\mathbb{C}}(\mathfrak{g}^*)$ denotes the complexification of $\wedge^k(\mathfrak{g}^*)$. We shall also denote by $d:\wedge^k_{\mathbb{C}}(\mathfrak{g}^*)\to\wedge^{k+1}_{\mathbb{C}}(\mathfrak{g}^*)$ the extension to $\wedge^k_{\mathbb{C}}(\mathfrak{g}^*)$ of the Chevalley–Eilenberg differential.

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie algebras endowed with complex structures $J$ and $J'$, respectively. A *complex isomorphism* between $(\mathfrak{g},J)$ and $(\mathfrak{g}',J')$ is an isomorphism $\alpha:\mathfrak{g}\to\mathfrak{g}'$ of Lie algebras such that $\alpha\circ J=J'\circ\alpha$. The latter condition is equivalent to say that $\alpha$, extended to the complexifications, preserves the bidegree.

It is clear that, if $\alpha:(\mathfrak{g},J)\to(\mathfrak{g}',J')$ is a complex isomorphism, then $\Omega'$ is a symplectic form on $\mathfrak{g}'$ compatible with $J'$ if and only if its pullback $\Omega=\alpha^*(\Omega')$ is a symplectic form on $\mathfrak{g}$ compatible with $J$. Therefore, the existence problem of compatible pairs is set up to complex isomorphisms.

Let $J$ be a complex structure on a nilpotent Lie algebra $\mathfrak{g}$ of real dimension 6. From Salamon's paper [8], this is equivalent to the existence of a basis $\{\omega_1, \omega_2, \omega_3\}$ for $\mathfrak{g}^{1,0} = \wedge^{1,0}(\mathfrak{g}^*)$ satisfying

$$d\omega_1 = 0, \qquad \omega_1 \wedge d\omega_2 = 0, \qquad \omega_1 \wedge \omega_2 \wedge d\omega_3 = 0.$$

These conditions imply that the expressions of $d\omega_1$, $d\omega_2$ and $d\omega_3$ in terms of $\{\omega_j, \bar{\omega}_j\}_{j=1}^3$ must have the form:

$$\begin{aligned} d\omega_1 &= 0,\\ d\omega_2 &= A_{12}\omega_1 \wedge \omega_2 + A_{13}\omega_1 \wedge \omega_3 + A_{1\bar{1}}\omega_1 \wedge \bar{\omega}_1 + A_{1\bar{2}}\omega_1 \wedge \bar{\omega}_2 + A_{1\bar{3}}\omega_1 \wedge \bar{\omega}_3,\\ d\omega_3 &= B_{12}\omega_1 \wedge \omega_2 + B_{13}\omega_1 \wedge \omega_3 + B_{1\bar{1}}\omega_1 \wedge \bar{\omega}_1 + B_{1\bar{2}}\omega_1 \wedge \bar{\omega}_2 + B_{1\bar{3}}\omega_1 \wedge \bar{\omega}_3\\ &\quad + B_{23}\omega_2 \wedge \omega_3 + B_{2\bar{1}}\omega_2 \wedge \bar{\omega}_1 + B_{2\bar{2}}\omega_2 \wedge \bar{\omega}_2 + B_{2\bar{3}}\omega_2 \wedge \bar{\omega}_3 \end{aligned} \tag{1}$$

for some complex coefficients $A$'s and $B$'s. Moreover, the complex structure $J$ is nilpotent if and only if there is a basis of $\mathfrak{g}^{1,0}$ such that all the coefficients $A_{12}$, $A_{13}$, $A_{1\bar{2}}$, $A_{1\bar{3}}$, $B_{13}$, $B_{1\bar{3}}$, $B_{23}$ and $B_{2\bar{3}}$ vanish (see Theorem 12 of [3]). In addition, $B_{12} = 0$ if and only if $J$ is abelian according to the previous definition.

Therefore, the nilpotent Lie algebras admitting a complex structure are determined by the structure equations (1) where the coefficients $A$'s and $B$'s must satisfy those compatibility conditions imposed by the Jacobi identity of the corresponding bracket (which is equivalent to requiring $d(d\omega_2) = d(d\omega_3) = 0$) and the nilpotency of the Lie algebra.

In Lemma 2.1, it is proved that, up to complex isomorphism, we can always suppose that the coefficients $A_{1\bar{2}}$, $B_{1\bar{3}}$, $B_{23}$ and $B_{2\bar{3}}$ in (1) vanish.

From now on, let us denote by $\{Z_1, Z_2, Z_3\}$ the basis for $\mathfrak{g}_{1,0}$ dual to $\{\omega_1, \omega_2, \omega_3\}$.

**Lemma 2.1.** *Let $J$ be a complex structure on a nilpotent Lie algebra $\mathfrak{g}$ of dimension 6. Then, the complex structure equations of $(\mathfrak{g}, J)$ can be expressed as*

$$\begin{aligned} d\omega_1 &= 0,\\ d\omega_2 &= A_{12}\omega_1 \wedge \omega_2 + A_{13}\omega_1 \wedge \omega_3 + A_{1\bar{1}}\omega_1 \wedge \bar{\omega}_1 + A_{1\bar{3}}\omega_1 \wedge \bar{\omega}_3,\\ d\omega_3 &= B_{12}\omega_1 \wedge \omega_2 + B_{13}\omega_1 \wedge \omega_3 + B_{1\bar{1}}\omega_1 \wedge \bar{\omega}_1 + B_{1\bar{2}}\omega_1 \wedge \bar{\omega}_2 + B_{2\bar{1}}\omega_2 \wedge \bar{\omega}_1\\ &\quad + B_{2\bar{2}}\omega_2 \wedge \bar{\omega}_2, \end{aligned}$$

*where the coefficients $A$'s and $B$'s are complex numbers satisfying the compatibility conditions imposed by the nilpotency of $\mathfrak{g}$ and the Jacobi identity of the bracket of $\mathfrak{g}$.*

**Proof.** Let us first see that the coefficient $B_{23}$ in Eq. (1) must be zero. In fact, if $B_{23}$ does not vanish then the $k$th bracket $[Z_2, \cdots [Z_2, [Z_2, Z_3]] \cdots] = (-B_{23})^k Z_3$ is nonzero for any $k \geq 1$, which is in contradiction to the nilpotency of $\mathfrak{g}$.

A similar argument, but now using the bracket $[Z_2, \bar{Z}_3] = -B_{2\bar{3}} Z_3$ and its complex conjugate $[\bar{Z}_2, Z_3] = -\bar{B}_{2\bar{3}}\bar{Z}_3$, leads to the fact that the nilpotency of $\mathfrak{g}$ implies $B_{2\bar{3}} = 0$.

Next we show that in Eq. (1) we can also consider that $B_{1\bar{3}}=0$. First of all, let us consider the complex transformation defined by

$$\omega_1'=\omega_1,\qquad \omega_2'=\omega_2,\qquad \omega_3'=\omega_3-\frac{P}{Q}\omega_2, \tag{2}$$

where $P, Q\in\mathbb{C}$, and $Q\neq 0$. From (1) with $B_{23}=B_{2\bar{3}}=0$, we get that with respect to the basis $\{\omega_1',\omega_2',\omega_3'\}$ for $\mathfrak{g}^{1,0}$ the structure equations become

$$\begin{aligned}
&\mathrm{d}\omega_1'=0,\\
&\mathrm{d}\omega_2'=A_{12}'\omega_1'\wedge\omega_2'+A_{13}'\omega_1'\wedge\omega_3'+A_{1\bar{1}}'\omega_1'\wedge\bar{\omega}_1'+A_{1\bar{2}}'\omega_1'\wedge\bar{\omega}_2'+A_{1\bar{3}}'\omega_1'\wedge\bar{\omega}_3',\\
&\mathrm{d}\omega_3'=B_{12}'\omega_1'\wedge\omega_2'+B_{13}'\omega_1'\wedge\omega_3'+B_{1\bar{1}}'\omega_1'\wedge\bar{\omega}_1'+B_{1\bar{2}}'\omega_1'\wedge\bar{\omega}_2'+B_{1\bar{3}}'\omega_1'\wedge\bar{\omega}_3'\\
&\qquad +B_{2\bar{1}}'\omega_2'\wedge\bar{\omega}_1'+B_{2\bar{2}}'\omega_2'\wedge\bar{\omega}_2',
\end{aligned}$$

where we denote by $A'$'s and $B'$'s the new coefficients. It is easy to check that

$$B_{1\bar{3}}'=B_{1\bar{3}}-A_{1\bar{3}}\frac{P}{Q}. \tag{3}$$

Now, if $B_{1\bar{3}}\neq 0$ and $A_{1\bar{3}}=0$ then from (1) we have $[Z_1,\bar{Z}_3]=-B_{1\bar{3}}Z_3$ and $[\bar{Z}_1,Z_3]=-\bar{B}_{1\bar{3}}\bar{Z}_3$, which imply that $Z_3,\bar{Z}_3\in\mathfrak{g}^k$ for any $k\geq 1$, and this is a contradiction to the nilpotency of $\mathfrak{g}$. Thus, if $B_{1\bar{3}}\neq 0$ then $A_{1\bar{3}}\neq 0$ and we can consider (2) with $P=B_{1\bar{3}}$ and $Q=A_{1\bar{3}}$. From (3) it follows that the new coefficient $B_{1\bar{3}}'=0$.

Therefore, we can suppose without loss of generality that the coefficients $B_{1\bar{3}}$, $B_{23}$ and $B_{2\bar{3}}$ in (1) vanish. Moreover, a direct calculation shows that the condition $\mathrm{d}^2\omega_2=0$ is equivalent to the following relations:[1]

$$\begin{array}{llll}
(12\bar{1})_2 & A_{13}B_{2\bar{1}}-A_{1\bar{3}}\bar{B}_{1\bar{2}}-|A_{1\bar{2}}|^2=0, & (1\bar{1}\bar{2})_2 & \bar{A}_{12}A_{1\bar{2}}+A_{1\bar{3}}\bar{B}_{12}=0,\\
(12\bar{2})_2 & A_{13}B_{2\bar{2}}-A_{1\bar{3}}\bar{B}_{2\bar{2}}=0, & (1\bar{1}\bar{3})_2 & \bar{A}_{13}A_{1\bar{2}}+A_{1\bar{3}}\bar{B}_{13}=0,\\
(13\bar{1})_2 & A_{1\bar{2}}\bar{A}_{1\bar{3}}=0. & &
\end{array}$$

If $A_{1\bar{2}}$ does not vanish then condition $(13\bar{1})_2$ implies $A_{1\bar{3}}=0$, and from $(1\bar{1}\bar{3})_2$ we also get $A_{13}=0$. But this is in contradiction to $(12\bar{1})_2$. Thus, the coefficient $A_{1\bar{2}}$ must be zero. □

In the next two propositions it is proved that either $B_{2\bar{2}}\neq 0$ or $B_{2\bar{2}}=A_{1\bar{3}}=0$ imply the nilpotency of the complex structure.

**Proposition 2.2.** *If in the structure equations given in* Lemma 2.1 *the coefficient* $B_{2\bar{2}}\neq 0$, *then the complex structure J is nilpotent.*

**Proof.** A direct calculation shows that the condition $\mathrm{d}^2\omega_3=0$ is equivalent to the following relations:

[1] In what follows, we shall use the labels $(jk\bar{l})_v$ and $(j\bar{k}\bar{l})_v$ for the coefficients of $\omega_j\wedge\omega_k\wedge\bar{\omega}_l$ and $\omega_j\wedge\bar{\omega}_k\wedge\bar{\omega}_l$, respectively, in the equation $\mathrm{d}^2\omega_v=0$, for $v=2,3$.

$(12\bar{1})_3$ $A_{12}B_{2\bar{1}} - \bar{A}_{1\bar{1}}B_{2\bar{2}} - B_{13}B_{2\bar{1}} = 0,$ $(1\bar{1}\bar{2})_3$ $A_{1\bar{1}}B_{2\bar{2}} - \bar{A}_{12}B_{1\bar{2}} = 0,$
$(12\bar{2})_3$ $A_{12}B_{2\bar{2}} - B_{13}B_{2\bar{2}} = 0,$ $(1\bar{1}\bar{3})_3$ $\bar{A}_{13}B_{1\bar{2}} + A_{1\bar{3}}B_{2\bar{1}} = 0,$
$(13\bar{1})_3$ $A_{13}B_{2\bar{1}} + \bar{A}_{1\bar{3}}B_{1\bar{2}} = 0,$ $(1\bar{2}\bar{3})_3$ $A_{1\bar{3}}B_{2\bar{2}} = 0,$
$(13\bar{2})_3$ $A_{13}B_{2\bar{2}} = 0,$ $(2\bar{1}\bar{2})_3$ $\bar{A}_{12}B_{2\bar{2}} = 0,$
$(23\bar{1})_3$ $\bar{A}_{1\bar{3}}B_{2\bar{2}} = 0,$ $(2\bar{1}\bar{3})_3$ $\bar{A}_{13}B_{2\bar{2}} = 0.$

From $(13\bar{2})_3$, $(23\bar{1})_3$ and $(2\bar{1}\bar{2})_3$ it follows that $A_{12} = A_{13} = A_{1\bar{3}} = 0$. Since $A_{12} = 0$ the equality $(12\bar{2})_3$ implies $B_{13} = 0$. Now, from $(12\bar{1})_3$ we deduce $A_{1\bar{1}} = 0$. Thus, the structure equations of $(\mathfrak{g}, J)$ are

$$
\begin{aligned}
&d\omega_1 = d\omega_2 = 0,\\
&d\omega_3 = B_{12}\omega_1\wedge\omega_2 + B_{1\bar{1}}\omega_1\wedge\bar{\omega}_1 + B_{1\bar{2}}\omega_1\wedge\bar{\omega}_2 + B_{2\bar{1}}\omega_2\wedge\bar{\omega}_1 + B_{2\bar{2}}\omega_2\wedge\bar{\omega}_2
\end{aligned}
$$

and therefore $J$ is nilpotent. □

The next result shows how the coefficients in the structure equations must be chosen in order to ensure the Jacobi identity of the corresponding bracket.

**Lemma 2.3.** *If $B_{2\bar{2}} = 0$ in Lemma 2.1 then, the Jacobi identity is satisfied if and only if*

$(12\bar{1})_2$ $A_{13}B_{2\bar{1}} - A_{1\bar{3}}\bar{B}_{1\bar{2}} = 0,$
$(1\bar{1}\bar{2})_2$ $A_{1\bar{3}}\bar{B}_{12} = 0,$
$(1\bar{1}\bar{3})_2$ $A_{1\bar{3}}\bar{B}_{13} = 0,$
$(12\bar{1})_3$ $A_{12}B_{2\bar{1}} - B_{13}B_{2\bar{1}} = 0,$
$(13\bar{1})_3$ $A_{13}B_{2\bar{1}} + \bar{A}_{1\bar{3}}B_{1\bar{2}} = 0,$
$(1\bar{1}\bar{2})_3$ $\bar{A}_{12}B_{1\bar{2}} = 0,$
$(1\bar{1}\bar{3})_3$ $\bar{A}_{13}B_{1\bar{2}} + A_{1\bar{3}}B_{2\bar{1}} = 0.$

**Proposition 2.4.** *If the coefficients $A_{1\bar{3}}$ and $B_{2\bar{2}}$ in the structure equations given in Lemma 2.1 vanish, then $J$ is nilpotent.*

**Proof.** First of all, since $A_{1\bar{3}} = 0$, if we proceed as in Lemma 2.1 and consider a complex transformation (2) with $P = B_{13}$ and $Q = A_{13}$, then we can suppose without loss of generality that $B_{13} = 0$.

On the other hand, the coefficients $A$'s and $B$'s in the structure equations must satisfy certain relations imposed by the nilpotency of $\mathfrak{g}$, together with the conditions of Lemma 2.3. But now, since $A_{1\bar{3}} = B_{2\bar{2}} = B_{13} = 0$, these conditions reduce to

$(12\bar{1})_2$ $A_{13}B_{2\bar{1}} = 0,$ $(1\bar{1}\bar{2})_3$ $\bar{A}_{1\bar{2}}B_{1\bar{2}} = 0,$
$(12\bar{1})_3$ $A_{12}B_{2\bar{1}} = 0,$ $(1\bar{1}\bar{3})_3$ $\bar{A}_{13}B_{1\bar{2}} = 0.$

If $B_{1\bar{2}} \neq 0$ or $B_{2\bar{1}} \neq 0$ then these equations imply $A_{12} = A_{13} = 0$, and thus the complex structure $J$ is clearly nilpotent.

Therefore, it remains to study the case $B_{1\bar{2}} = B_{2\bar{1}} = 0$, so next we will restrict our attention to structure equations of the form:

$$\begin{aligned} &\mathrm{d}\omega_1 = 0, \qquad \mathrm{d}\omega_2 = A_{12}\omega_1 \wedge \omega_2 + A_{13}\omega_1 \wedge \omega_3 + A_{1\bar{1}}\omega_1 \wedge \bar{\omega}_1, \\ &\mathrm{d}\omega_3 = B_{12}\omega_1 \wedge \omega_2 + B_{1\bar{1}}\omega_1 \wedge \bar{\omega}_1 \end{aligned} \tag{4}$$

and we shall show that in this case the nilpotency of the Lie algebra $\mathfrak{g}$ implies the nilpotency of the complex structure $J$.

We consider two cases depending on the vanishing of the coefficient $A_{13}$ in (4). If $A_{13} = 0$ then $A_{12} = 0$, because otherwise from (4) it would follow that the $k$th Lie bracket:

$$[Z_1, \cdots [Z_1, [Z_1, Z_2]] \cdots ] = (-1)^k (A_{12}^k Z_2 + B_{12} A_{12}^{k-1} Z_3) \neq 0$$

for any $k$, which is in contradiction with the nilpotency of $\mathfrak{g}$. But if $A_{13} = A_{12} = 0$ then $J$ is nilpotent.

Next, let us suppose $A_{13} \neq 0$. In this case, by multiplying $\omega_2$ by $1/A_{13}$, we can suppose $A_{13} = 1$. From (4) we get the following brackets:

$$\begin{aligned} [Z_1, Z_3] &= -Z_2, \\ [Z_1, [Z_1, Z_3]] &= A_{12} Z_2 + B_{12} Z_3, \\ [Z_1, [Z_1, [Z_1, Z_3]]] &= -(A_{12}^2 + B_{12}) Z_2 - A_{12} B_{12} Z_3, \\ [Z_1, [Z_1, [Z_1, [Z_1, Z_3]]]] &= A_{12}(A_{12}^2 + 2B_{12}) Z_2 + B_{12}(A_{12}^2 + B_{12}) Z_3, \\ [Z_1, [Z_1, [Z_1, [Z_1, [Z_1, Z_3]]]]] &= -(A_{12}^4 + 3A_{12}^2 B_{12} + B_{12}^2) Z_2 - A_{12} B_{12}(A_{12}^2 + 2B_{12}) Z_3. \end{aligned}$$

We prove now that if $B_{12} \neq 0$ then the coefficients of $Z_2$ and $Z_3$ in the latter bracket do not vanish simultaneously. The coefficient of $Z_3$ vanishes if $A_{12} = 0$ or $A_{12}^2 = -2B_{12}$; but if $A_{12} = 0$ then the coefficient of $Z_2$ is nonzero, and if $A_{12}^2 = -2B_{12}$ then $[Z_1, [Z_1, [Z_1, [Z_1, [Z_1, Z_3]]]]] = B_{12}^2 Z_2 \neq 0$.

Therefore, if $B_{12}$ does not vanish then $\mathfrak{g}^5$ is not zero, which is in contradiction to the fact that any six-dimensional nilpotent Lie algebra $\mathfrak{g}$ has step of nilpotency $\leq 5$. Thus, if $A_{13} \neq 0$ then $B_{12} = 0$. Moreover, $A_{12} = 0$ because otherwise $\mathfrak{g}^5$ would be again nonzero. So Eq. (4) reduce to

$$\mathrm{d}\omega_1 = 0, \qquad \mathrm{d}\omega_2 = A_{13}\omega_1 \wedge \omega_3 + A_{1\bar{1}}\omega_1 \wedge \bar{\omega}_1, \qquad \mathrm{d}\omega_3 = B_{1\bar{1}}\omega_1 \wedge \bar{\omega}_1$$

and, interchanging $\omega_2$ with $\omega_3$, we get again the equations of a nilpotent complex structure. □

Let $\mathfrak{g}$ be a nilpotent Lie algebra having a complex structure $J$, and let us denote by $\mathcal{S}_c(\mathfrak{g}, J)$ the set of all symplectic forms $\Omega$ on $\mathfrak{g}$ which are compatible with $J$. Then $\mathcal{S}_c(\mathfrak{g}, J)$ can be identified with

$$\mathcal{S}_c(\mathfrak{g}, J) = \tilde{Z}^{1,1}(\mathfrak{g}, J) \cap \mathcal{V}(\mathfrak{g}), \tag{5}$$

where $\tilde{Z}^{1,1}(\mathfrak{g}, J)$ is the vector space of all closed real $(1, 1)$-forms on $\mathfrak{g}^{\mathbb{C}}$ and $\mathcal{V}(\mathfrak{g})$ is the set of 2-forms on $\mathfrak{g}$ which are nondegenerate. Moreover, it is clear that $\tilde{Z}^{1,1}(\mathfrak{g}, J)$ is identified with $Z^{1,1}(\mathfrak{g}, J) \cap \{\Omega \in \wedge^{1,1}(\mathfrak{g}^*) | \Omega = \bar{\Omega}\}$, where $Z^{1,1}(\mathfrak{g}, J) = \ker\{d|_{\wedge^{1,1}(\mathfrak{g}^*)} : \wedge^{1,1}(\mathfrak{g}^*) \to \wedge^3_{\mathbb{C}}(\mathfrak{g}^*)\}$.

Therefore, the set $\mathcal{S}_c(\mathfrak{g}, J)$ is an open subset of the vector space $\tilde{Z}^{1,1}(\mathfrak{g}, J)$, which eventually could be empty. If there exist compatible symplectic forms, then the dimension of $\mathcal{S}_c(\mathfrak{g}, J)$ is equal to $\dim \tilde{Z}^{1,1}(\mathfrak{g}, J)$, because the tangent space $T_\Omega \mathcal{S}_c(\mathfrak{g}, J)$ at any $J$-compatible form $\Omega$ is identified to $\tilde{Z}^{1,1}(\mathfrak{g}, J)$.

Notice that $\mathfrak{g}$ has a pseudo-Kähler metric if and only if $\mathcal{S}_c(\mathfrak{g}, J) \neq \emptyset$, for some complex structure $J$ on $\mathfrak{g}$. So, the nonexistence of pseudo-Kähler metrics on $\mathfrak{g}$ is a subtle problem, because in order to prove that there is no compatible pair $(J, \Omega)$ we must take into account the whole of the set $\mathcal{C}(\mathfrak{g})$ of complex structures $J$ on $\mathfrak{g}$ and prove the nonexistence of a compatible $\Omega$ for any $J$. In addition, there are nilpotent Lie algebras having complex structures $J_1$ and $J_2$ such that there exist compatible symplectic forms for $J_1$ but with $\mathcal{S}_c(\mathfrak{g}, J_2) = \emptyset$.

In the following theorem we prove that, in dimension 6, $\mathcal{S}_c(\mathfrak{g}, J) = \emptyset$ for any complex structure $J$ that is not nilpotent.

**Theorem 2.5.** *The complex structure underlying any pseudo-Kähler metric on a six-dimensional nilpotent Lie algebra is nilpotent.*

**Proof.** Let $J$ be a complex structure on a six-dimensional nilpotent Lie algebra $\mathfrak{g}$. It is sufficient to prove that if $J$ is not nilpotent then any closed 2-form compatible with $J$ is degenerate. In view of Propositions 2.2 and 2.4, a nonnilpotent complex structure can be obtained only when $B_{2\bar{2}} = 0$ and $A_{1\bar{3}} \neq 0$. Let us suppose then that the structure equations for $(\mathfrak{g}, J)$ are

$$\begin{aligned} d\omega_1 &= 0, \\ d\omega_2 &= A_{12}\omega_1 \wedge \omega_2 + A_{13}\omega_1 \wedge \omega_3 + A_{1\bar{1}}\omega_1 \wedge \bar{\omega}_1 + A_{1\bar{3}}\omega_1 \wedge \bar{\omega}_3, \\ d\omega_3 &= B_{12}\omega_1 \wedge \omega_2 + B_{13}\omega_1 \wedge \omega_3 + B_{1\bar{1}}\omega_1 \wedge \bar{\omega}_1 + B_{1\bar{2}}\omega_1 \wedge \bar{\omega}_2 + B_{2\bar{1}}\omega_2 \wedge \bar{\omega}_1, \end{aligned}$$

where the coefficients must guarantee the nilpotency of $\mathfrak{g}$ and satisfy the relations given in Lemma 2.3. From $(1\bar{1}\bar{2})_2$ in Lemma 2.3 it follows that $B_{12} = 0$, and from condition $(1\bar{1}\bar{3})_2$ we deduce that $B_{13} = 0$. Moreover, the nilpotency of $\mathfrak{g}$ implies that $A_{12} = 0$, because otherwise $[Z_1, \cdots [Z_1, [Z_1, Z_2]] \cdots ] = (-A_{12})^k Z_2 \neq 0$ for any $k$.

Let $\Omega$ be a 2-form compatible with $J$, that is, $\Omega \in \wedge^{1,1}(\mathfrak{g})$. Therefore,

$$\Omega = \sum_{j=1}^{3} (a_j\omega_1 + b_j\omega_2 + c_j\omega_3) \wedge \bar{\omega}_j$$

for some coefficients $a_j, b_j, c_j \in \mathbb{C}$, for $j = 1, 2, 3$. A direct calculation shows that $\Omega \in Z^{1,1}(\mathfrak{g}, J)$, i.e. $d\Omega = 0$, implies the following relations:[2]

$$\begin{aligned} &(12\bar{1})_0 \quad a_3\bar{B}_{1\bar{2}} - b_2\bar{A}_{1\bar{1}} - b_3\bar{B}_{1\bar{1}} = 0, &&(13\bar{2})_0 \quad b_2A_{13} - c_3\bar{B}_{2\bar{1}} = 0, \\ &(12\bar{2})_0 \quad b_3\bar{B}_{2\bar{1}} = 0, &&(23\bar{1})_0 \quad b_2\bar{A}_{1\bar{3}} - c_3\bar{B}_{1\bar{2}} = 0. \end{aligned}$$

[2] Here we use the label $(jk\bar{l})_0$ for the coefficient of $\omega_j \wedge \omega_k \wedge \bar{\omega}_l$ in the equation $d\Omega = 0$.

Notice that if $\bar{A}_{1\bar{3}}\bar{B}_{2\bar{1}} - A_{13}\bar{B}_{1\bar{2}} = 0$, then $(1\bar{1}\bar{3})_3$ in Lemma 2.3 implies $B_{2\bar{1}} = 0$ (because $A_{1\bar{3}} \neq 0$), and thus $B_{1\bar{2}} = 0$ by condition $(12\bar{1})_2$. Therefore, in this case the complex structure $J$ is nilpotent (it suffices to interchange $\omega_2$ with $\omega_3$ in the equations above).

Since $J$ is nonnilpotent, we have necessarily

$$\det\begin{pmatrix} A_{13} & -\bar{B}_{2\bar{1}} \\ \bar{A}_{1\bar{3}} & -\bar{B}_{1\bar{2}} \end{pmatrix} \neq 0,$$

so from $(13\bar{2})_0$ and $(23\bar{1})_0$, we conclude that $b_2 = c_3 = 0$ if the 2-form $\Omega$ is closed.

Moreover, arguing as above, equations in Lemma 2.3 imply that $B_{2\bar{1}} = 0$ if and only if $B_{1\bar{2}} = 0$, and in this case $J$ is nilpotent. Therefore, $B_{1\bar{2}}B_{2\bar{1}} \neq 0$ and the conditions $(12\bar{1})_0$ and $(12\bar{2})_0$ are satisfied if and only if $a_3 = b_3 = 0$.

Finally, since $a_3 = b_3 = c_3 = 0$, it is clear that $\Omega^3 = 0$, that is, $\Omega$ must be degenerate if it is closed; thus, $Z^{1,1}(\mathfrak{g}, J) \cap \mathcal{V}(\mathfrak{g}) = \emptyset$ for any nonnilpotent complex structure $J$ on $\mathfrak{g}$. □

**Corollary 2.6.** *In dimension 6, if J has a compatible symplectic form then J is nilpotent.*

We conjecture that this result is still true for any dimension $2n$, that is: any complex structure on a nilpotent Lie algebra of dimension $2n$ must be nilpotent in presence of a compatible symplectic form. Corollary 2.6 together with Remark 4.3, give an affirmative answer for $n \leq 3$.

**Remark 2.7.** Notice that in the proof of Theorem 2.5, we have obtained the following slightly stronger result: If a complex structure $J$ possesses a compatible closed 2-form (not necessarily real) then $J$ must be nilpotent.

Following the notation below, the Lie algebra $\mathfrak{h}_{26} = (0, 0, 12, 13, 23, 14 + 25)$ has symplectic forms and complex structures [8] and, since the centre of $\mathfrak{h}_{26}$ is one-dimensional, any complex structure cannot be nilpotent [3]. From Corollary 2.6, given any symplectic form $\Omega$ on $\mathfrak{h}_{26}$, there is no complex structure $J$ on $\mathfrak{h}_{26}$ compatible with $\Omega$, that is as in the following corollary.

**Corollary 2.8.** *Any Lie algebra $\mathfrak{g}$ isomorphic to $\mathfrak{h}_{26} = (0, 0, 12, 13, 23, 14 + 25)$ has complex and symplectic structures, but there exists no pseudo-Kähler metric on $\mathfrak{g}$, i.e. $\mathcal{S}_c(\mathfrak{g}, J) = \emptyset$ for any $J \in \mathcal{C}(\mathfrak{g})$.*

As far as we know, this is the first known example of a nilpotent Lie algebra having such a property.

## 3. The classification

In this section, we classify six-dimensional nilpotent Lie algebras admitting pseudo-Kähler metrics. More precisely, we shall prove the following result.

**Theorem 3.1.** *Let $\mathfrak{g}$ be a (nonabelian) nilpotent Lie algebra of dimension 6. Then, $\mathfrak{g}$ possesses a compatible pair $(J, \Omega)$ if and only if $\mathfrak{g}$ is isomorphic to one of the following Lie algebras*:

$$\begin{aligned}
\mathfrak{h}_2 &= (0, 0, 0, 0, 12, 34),\\
\mathfrak{h}_4 &= (0, 0, 0, 0, 12, 14 + 23),\\
\mathfrak{h}_5 &= (0, 0, 0, 0, 13 + 42, 14 + 23),\\
\mathfrak{h}_6 &= (0, 0, 0, 0, 12, 13),\\
\mathfrak{h}_7 &= (0, 0, 0, 12, 13, 23),\\
\mathfrak{h}_8 &= (0, 0, 0, 0, 0, 12),\\
\mathfrak{h}_9 &= (0, 0, 0, 0, 12, 14 + 25),\\
\mathfrak{h}_{10} &= (0, 0, 0, 12, 13, 14),\\
\mathfrak{h}_{11} &= (0, 0, 0, 12, 13, 14 + 23),\\
\mathfrak{h}_{12} &= (0, 0, 0, 12, 13, 24),\\
\mathfrak{h}_{13} &= (0, 0, 0, 12, 13 + 14, 24),\\
\mathfrak{h}_{14} &= (0, 0, 0, 12, 14, 13 + 42),\\
\mathfrak{h}_{15} &= (0, 0, 0, 12, 13 + 42, 14 + 23).
\end{aligned}$$

*Notation*. Some explanation about this notation is needed. In the list above, we have combined the notation $\mathfrak{h}_k$ of the table given in [2] and the structure description of the Lie algebras as it appears in [8]. For example, $\mathfrak{h}_2 = (0, 0, 0, 0, 12, 34)$ means that there is a basis $\{X_1, \dots, X_6\}$ for the Lie algebra in terms of which the only nonzero bracket relations are $[X_1, X_2] = -X_5$ and $[X_3, X_4] = -X_6$. Equivalently, in terms of the dual basis $\{\alpha_1, \dots, \alpha_6\}$ the Chevalley–Eilenberg differential d is given by

$$\mathrm{d}\alpha_1 = \mathrm{d}\alpha_2 = \mathrm{d}\alpha_3 = \mathrm{d}\alpha_4 = 0, \qquad \mathrm{d}\alpha_5 = \alpha_1 \wedge \alpha_2, \qquad \mathrm{d}\alpha_6 = \alpha_3 \wedge \alpha_4.$$

For more information on the ascending series, Betti numbers or the dimension of the set of symplectic forms on each Lie algebra, see [2,8].

It follows from [2] that any Lie algebra admitting a nilpotent complex structure must be isomorphic to $\mathfrak{h}_k$, for some $k \leq 16$. On the other hand, among these algebras, only $\mathfrak{h}_3 = (0, 0, 0, 0, 0, 12+34)$ and $\mathfrak{h}_{16} = (0, 0, 0, 12, 14, 24)$ do not possess symplectic forms. Therefore, Theorem 3.1 states that these two are the only algebras not having pseudo-Kähler metrics. Thus, the classification result can be summarized as follows.

**Corollary 3.2.** *In dimension 6, the Lie algebra $\mathfrak{g}$ has compatible pairs $(J, \Omega)$ if and only if it admits both symplectic and nilpotent complex structures.*

**Remark 3.3.** In [5], we have classified six-dimensional nilpotent Lie algebras having pseudo-Kähler metrics whose underlying complex structure is abelian, so Theorem 3.1 extends our previous results. Also, it completes some partial results on the existence of pseudo-Kähler metrics given in [6, p. 18].

Our proof of Theorem 3.1 will consist in a case by case study, in which we construct explicit pseudo-Kähler metrics when they exist. This is detailed in Propositions 3.5–3.10.

First of all, by Theorem 2.5 it suffices to restrict our attention to nilpotent complex structures in order to obtain such classification. Thus, along this section, $J$ will denote a *nilpotent* complex structure on a nilpotent Lie algebra $\mathfrak{g}$ of dimension 6. Therefore, the (complex) structure equations of $(\mathfrak{g}, J)$ can always be expressed as

$$\begin{aligned} d\omega_1 &= 0, \qquad d\omega_2 = A_{1\bar{1}}\omega_1 \wedge \bar{\omega}_1, \\ d\omega_3 &= B_{12}\omega_1 \wedge \omega_2 + B_{1\bar{1}}\omega_1 \wedge \bar{\omega}_1 + B_{1\bar{2}}\omega_1 \wedge \bar{\omega}_2 + B_{2\bar{1}}\omega_2 \wedge \bar{\omega}_1 + B_{2\bar{2}}\omega_2 \wedge \bar{\omega}_2, \end{aligned}$$

where the coefficients $A$'s and $B$'s are complex numbers satisfying those restrictions imposed by the Jacobi identity of the Lie bracket of $\mathfrak{g}$. Observe that the nilpotency of $\mathfrak{g}$ follows from the form of these equations. Also, notice that $J$ is abelian if and only if $B_{12} = 0$.

Let $\Omega$ be a 2-form on $\mathfrak{g}$ of type $(1, 1)$ with respect to $J$. Then

$$\Omega = \sum_{j=1}^{3} (a_j\omega_1 + b_j\omega_2 + c_j\omega_3) \wedge \bar{\omega}_j,$$

where $a_j, b_j, c_j \in \mathbb{C}$. It is clear that $\Omega$ is a real form, i.e. $\Omega = \bar{\Omega}$, if and only if these coefficients satisfy $b_1 = -\bar{a}_2$, $c_1 = -\bar{a}_3$, $c_2 = -\bar{b}_3$, and $a_1 + \bar{a}_1 = b_2 + \bar{b}_2 = c_3 + \bar{c}_3 = 0$, that is, $a_1, b_2, c_3$ are purely imaginary.

**Lemma 3.4.** *Let J be a nilpotent complex structure on* $\mathfrak{g}$, *with structure equations as above. Denote by* $V(\mathfrak{g}, J)$ *the real vector space of all* $(\mathrm{i}a_1, \mathrm{i}b_2, \mathrm{i}c_3, a_2, a_3, b_3) \in \mathbb{R}^3 \times \mathbb{C}^3$ *satisfying the conditions*:

$$\begin{array}{ll} (12\bar{1})_0 & a_3\bar{B}_{1\bar{2}} - b_2\bar{A}_{1\bar{1}} - b_3\bar{B}_{1\bar{1}} - \bar{a}_3 B_{12} = 0, \\ (12\bar{2})_0 & a_3\bar{B}_{2\bar{2}} - b_3\bar{B}_{2\bar{1}} - \bar{b}_3 B_{12} = 0, \\ (12\bar{3})_0 & c_3 B_{12} = 0, \\ (13\bar{1})_0 & \bar{b}_3\bar{A}_{1\bar{1}} - c_3\bar{B}_{1\bar{1}} = 0, \\ (13\bar{2})_0 & c_3\bar{B}_{2\bar{1}} = 0, \\ (23\bar{1})_0 & c_3\bar{B}_{1\bar{2}} = 0, \\ (23\bar{2})_0 & c_3\bar{B}_{2\bar{2}} = 0. \end{array} \qquad (6)$$

*If the set* $\mathcal{S}_c(\mathfrak{g}, J)$ *of compatible symplectic forms is nonempty, then* $\dim \mathcal{S}_c(\mathfrak{g}, J) = \dim V(\mathfrak{g}, J)$.

**Proof.** Since $\dim \mathcal{S}_c(\mathfrak{g}, J) = \dim \tilde{Z}^{1,1}(\mathfrak{g}, J)$, it suffices to check that the real vector spaces $\tilde{Z}^{1,1}(\mathfrak{g}, J)$ and $V(\mathfrak{g}, J)$ have the same dimension. But a simple calculation using the structure equations of $J$ above shows that a real $(1, 1)$-form $\Omega$ is closed if and only if conditions (6) hold. □

On the other hand, the nondegeneration of $\Omega$, i.e. $\Omega^3 \neq 0$, is equivalent to

$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ -\bar{a}_2 & b_2 & b_3 \\ -\bar{a}_3 & -\bar{b}_3 & c_3 \end{bmatrix} \neq 0. \qquad (7)$$

Notice that if $J$ and $\Omega$ are compatible, then we can define a pseudo-Kähler metric $g$ by $g(X,Y) = \Omega(X,JY)$, which in terms of the 6-tuple $(a_1,a_2,a_3,b_2,b_3,c_3)$ and the basis $\{\omega_j,\bar\omega_j\}_{j=1}^3$ is given by

$$\begin{aligned} g &= -\mathrm{i}(a_1\omega_1\#\bar\omega_1 + b_2\omega_2\#\bar\omega_2 + c_3\omega_3\#\bar\omega_3 + a_2\omega_1\#\bar\omega_2 - \bar a_2\omega_2\#\bar\omega_1 + a_3\omega_1\#\bar\omega_3 \\ &\quad - \bar a_3\omega_3\#\bar\omega_1 + b_3\omega_2\#\bar\omega_3 - \bar b_3\omega_3\#\bar\omega_2), \end{aligned} \tag{8}$$

where # denotes the symmetric product.

**Proposition 3.5.** *There are pseudo-Kähler metrics on* $\mathfrak{h}_2$, $\mathfrak{h}_4$ *and* $\mathfrak{h}_6$. *More precisely*:

- (i) *The Lie algebra* $\mathfrak{h}_2 = (0,0,0,0,12,34)$ *has a complex structure J such that its set of compatible symplectic forms* $\mathcal{S}_c(\mathfrak{h}_2, J)$ *is six-dimensional*.
- (ii) *There are complex structures J on* $\mathfrak{h}_4 = (0,0,0,0,12,14+23)$ *and* $\mathfrak{h}_6 = (0,0,0,0,12,13)$ *whose sets of compatible symplectic forms have dimension 5*.

**Proof.** Let us consider complex equations of the form:

$$\mathrm{d}\omega_1 = \mathrm{d}\omega_2 = 0, \qquad \mathrm{d}\omega_3 = \omega_1\wedge\omega_2 + \omega_1\wedge\bar\omega_2 + B_{2\bar1}\omega_2\wedge\bar\omega_1. \tag{9}$$

First, we shall see that these equations define a complex structure $J$ on $\mathfrak{h}_2$ when $B_{2\bar1} = 1$. In fact, if we write $\omega_1 = \beta_1 + \mathrm{i}\beta_2$, $\omega_2 = \beta_3 + \mathrm{i}\beta_4$ and $\omega_3 = \beta_5 + \mathrm{i}\beta_6$, then the (real) structure equations of the underlying Lie algebra are

$$\begin{aligned} &\mathrm{d}\beta_1 = \mathrm{d}\beta_2 = \mathrm{d}\beta_3 = \mathrm{d}\beta_4 = 0, \qquad \mathrm{d}\beta_5 = \beta_1\wedge\beta_3 - \beta_2\wedge\beta_4, \\ &\mathrm{d}\beta_6 = -\beta_1\wedge\beta_4 + 3\beta_2\wedge\beta_3. \end{aligned}$$

Since $\mathrm{d}(-\sqrt3\beta_5+\beta_6) = (-\beta_1+\sqrt3\beta_2)\wedge(\sqrt3\beta_3+\beta_4)$, and $\mathrm{d}(-\sqrt3\beta_5-\beta_6) = (-\beta_1-\sqrt3\beta_2)\wedge(\sqrt3\beta_3-\beta_4)$, if we consider the (real) transformation:

$$\begin{aligned} &\alpha_1 = -\beta_1+\sqrt3\beta_2, \qquad \alpha_2 = \sqrt3\beta_3+\beta_4, \quad \alpha_3 = -\beta_1-\sqrt3\beta_2, \\ &\alpha_4 = \sqrt3\beta_3-\beta_4, \qquad \alpha_5 = -\sqrt3\beta_5+\beta_6, \qquad \alpha_6 = -\sqrt3\beta_5-\beta_6, \end{aligned}$$

then $\mathrm{d}\alpha_1 = \mathrm{d}\alpha_2 = \mathrm{d}\alpha_3 = \mathrm{d}\alpha_4 = 0$, $\mathrm{d}\alpha_5 = \alpha_1\wedge\alpha_2$, $\mathrm{d}\alpha_6 = \alpha_3\wedge\alpha_4$, equations which correspond to the Lie algebra $\mathfrak{h}_2$. On the other hand, from (6) and (9) it follows that $\Omega$ is closed if and only if $a_3 - \bar a_3 = 0$, $b_3 + \bar b_3 = 0$ and $c_3 = 0$. Moreover, $\Omega$ is nondegenerate if and only if $a_3b_3(a_2+\bar a_2) \neq |a_3|^2b_2 + |b_3|^2a_1$. Therefore, $\mathcal{S}_c(\mathfrak{h}_2, J)$ is nonempty and Lemma 3.4 implies that $\dim \mathcal{S}_c(\mathfrak{h}_2, J) = 6$. This completes the proof of (i).

Now, if $B_{2\bar1} = 2$ in Eq. (9) and we define $\alpha_1,\dots,\alpha_6$ by $\alpha_4 + \mathrm{i}\alpha_2 = \omega_1$, $\alpha_3 + 2\mathrm{i}\alpha_1 = 4\omega_2$ and $\alpha_5 + \mathrm{i}\alpha_6 = \omega_3$, then a simple calculation shows that $\alpha_1,\dots,\alpha_4$ are closed, $\mathrm{d}\alpha_5 = \alpha_1\wedge\alpha_2$ and $\mathrm{d}\alpha_6 = \alpha_1\wedge\alpha_4 + \alpha_2\wedge\alpha_3$. Therefore, the equations above define a complex structure $J$ on $\mathfrak{h}_4$ when $B_{2\bar1} = 2$. Moreover, from (6) we have that $\Omega$ is closed if and only if $c_3 = 0$, $a_3 = \bar a_3$ and $\bar b_3 = -2b_3$, which implies $b_3 = 0$. The nondegeneration of $\Omega$ is equivalent to $|a_3|^2b_2 \neq 0$. Thus, $\mathcal{S}_c(\mathfrak{h}_4, J) \neq \emptyset$ and $\dim \mathcal{S}_c(\mathfrak{h}_4, J) = 5$ by Lemma 3.4.

Finally, let us see that Eq. (9) with $B_{2\bar1} = 0$ define a complex structure $J$ on $\mathfrak{h}_6$; in fact, if we consider $\alpha_1,\dots,\alpha_6$ given by $\alpha_2 + \mathrm{i}\alpha_3 = \omega_1$, $\alpha_1 + \mathrm{i}\alpha_4 = -2\omega_2$ and $\alpha_5 + \mathrm{i}\alpha_6 = \omega_3$, then it is easy to check that $\mathfrak{h}_6$ is the underlying Lie algebra on which $J$ is defined. Moreover, since it follows from (6) and (7) that a compatible 2-form $\Omega$ is closed and nondegenerate if and only if $b_3 = c_3 = 0$, $a_3 = \bar{a}_3$ and $|a_3|^2 b_2 \neq 0$, this implies that $\mathcal{S}_c(\mathfrak{h}_6, J)$ has dimension 5.□

It is worthy to remark that $\mathfrak{h}_2$ and $\mathfrak{h}_4$ admit abelian complex structures, but none of them admits compatible symplectic form (see [5] for details).

**Proposition 3.6.** *There are abelian complex structures J on* $\mathfrak{h}_5 = (0, 0, 0, 0, 13+42, 14+23)$ *having compatible symplectic forms. Moreover, any such J satisfies* $\dim \mathcal{S}_c(\mathfrak{h}_5, J) = 6$.

**Proof.** In [5], it is proved that any abelian complex structure on $\mathfrak{h}_5$ having compatible $\Omega$ can be expressed, up to complex transformation, by the structure equations:

$$\mathrm{d}\omega_1 = \mathrm{d}\omega_2 = 0, \qquad \mathrm{d}\omega_3 = \omega_1 \wedge \bar{\omega}_2.$$

Now (6) implies that $\mathrm{d}\Omega = 0$ if and only if $a_3 = c_3 = 0$. Moreover, $\Omega^3 \neq 0$ is equivalent to $|b_3|^2 a_1 \neq 0$. From Lemma 3.4 it follows that the set $\mathcal{S}_c(\mathfrak{h}_5, J)$ is six-dimensional for any such abelian $J$. □

Notice that $\mathfrak{h}_5$ is the Lie algebra underlying the Iwasawa nilmanifold, so this algebra has a complex structure $J$ such that $[JX, Y] = J[X, Y]$ for all $X, Y \in \mathfrak{h}_5$. In [4] it is proved that $\mathcal{S}_c(\mathfrak{g}, J) = \emptyset$ for any complex structure $J$ satisfying this condition on a nilpotent Lie algebra $\mathfrak{g}$.

It is worthy to remark that $\mathfrak{h}_5$ has also abelian complex structures which do not possess compatible symplectic form [5].

**Proposition 3.7.** *Any complex structure J on* $\mathfrak{h}_8 = (0, 0, 0, 0, 0, 12)$ *is abelian, and the set of compatible symplectic forms has dimension 6 for any such J.*

**Proof.** Since the first Betti number of $\mathfrak{h}_8$ is 5, any complex structure must be abelian, an observation already made in [8] (see also Proposition 5.2 in [4]). Moreover, it is proved in [5] that, up to a complex transformation, the structure equations of any $J$ on $\mathfrak{h}_8$ reduce to

$$\mathrm{d}\omega_1 = \mathrm{d}\omega_2 = 0, \qquad \mathrm{d}\omega_3 = \omega_1 \wedge \bar{\omega}_1.$$

So, a compatible form $\Omega$ is closed if and only if $b_3 = c_3 = 0$, and the nondegeneration of $\Omega$ is equivalent to $b_2|a_3|^2 \neq 0$; this implies that $\mathcal{S}_c(\mathfrak{h}_8, J)$ is six-dimensional by Lemma 3.4. □

**Proposition 3.8.** *Any complex structure J on* $\mathfrak{h}_9 = (0, 0, 0, 0, 12, 14 + 25)$ *is abelian and has a compatible symplectic form. Moreover,* $\dim \mathcal{S}_c(\mathfrak{h}_9, J) = 4$ *for any J.*

**Proof.** From the proof of Theorem 2.5, it is easy to see that a nonnilpotent complex structure cannot exist on a six-dimensional nilpotent Lie algebra whose first Betti number equals 4. Therefore, any complex structure $J$ on $\mathfrak{h}_9$ is necessarily nilpotent, because $\mathfrak{h}_9$ has first Betti number equal to 4. Moreover, in [4] (Proposition 5.3) it is proved that any nilpotent complex structure $J$ on $\mathfrak{h}_9$ is abelian, and in [5] it is shown that, up to a complex transformation, the structure equations of any $J$ on $\mathfrak{h}_9$ are

$$d\omega_1 = 0, \qquad d\omega_2 = \omega_1 \wedge \bar{\omega}_1, \qquad d\omega_3 = \omega_1 \wedge \bar{\omega}_2 + B_{2\bar{1}}\omega_2 \wedge \bar{\omega}_1,$$

where $|B_{2\bar{1}}| = 1$. Now, any real 2-form $\Omega$ compatible with $J$ is closed if and only if $b_3 = c_3 = 0$ and $a_3 = b_2$. Moreover, $\Omega$ is nondegenerate if and only if $b_2 \neq 0$. Thus, for any $J$ on $\mathfrak{h}_9$, the space of compatible symplectic forms is four-dimensional. □

**Proposition 3.9.** *The Lie algebras* $\mathfrak{h}_7$ *and* $\mathfrak{h}_{10}, \dots, \mathfrak{h}_{14}$ *admit pseudo-Kähler metrics. More precisely*:

(i) *The Lie algebras* $\mathfrak{h}_7 = (0, 0, 0, 12, 13, 23)$ *and* $\mathfrak{h}_{14} = (0, 0, 0, 12, 14, 13 + 42)$ *have a complex structure admitting a five-dimensional set of compatible symplectic forms.*

(ii) *The Lie algebras* $\mathfrak{h}_{10} = (0, 0, 0, 12, 13, 14)$, $\mathfrak{h}_{11} = (0, 0, 0, 12, 13, 14 + 23)$, $\mathfrak{h}_{12} = (0, 0, 0, 12, 13, 24)$ *and* $\mathfrak{h}_{13} = (0, 0, 0, 12, 13 + 14, 24)$ *possess a complex structure having a four-dimensional set of compatible symplectic forms.*

**Proof.** Let us consider the complex equations:

$$d\omega_1 = 0, \qquad d\omega_2 = \omega_1 \wedge \bar{\omega}_1, \qquad d\omega_3 = \omega_1 \wedge \omega_2 + B_{1\bar{2}}\omega_1 \wedge \bar{\omega}_2 + B_{2\bar{1}}\omega_2 \wedge \bar{\omega}_1,$$

where $B_{1\bar{2}} = r + \mathrm{i}t$, $B_{2\bar{1}} = s - \mathrm{i}t$, for $r, s, t \in \mathbb{R}$, that is, $\mathrm{Im}\, B_{2\bar{1}} = -\mathrm{Im}\, B_{1\bar{2}}$. Now, we define $\alpha_1, \dots, \alpha_6$ by $\alpha_1 + \mathrm{i}\alpha_2 = \omega_1$, $-2\alpha_3 - 2\mathrm{i}\alpha_4 = \omega_2$ and $-4\alpha_5 - 4\mathrm{i}\alpha_6 = \omega_3$. A simple calculation shows that

$$\begin{aligned} d\alpha_1 &= d\alpha_2 = d\alpha_3 = 0, \qquad d\alpha_4 = \alpha_1 \wedge \alpha_2, \\ d\alpha_5 &= \frac{1 + r - s}{2}\alpha_1 \wedge \alpha_3 - \frac{1 - r + s}{2}\alpha_2 \wedge \alpha_4, \\ d\alpha_6 &= t\alpha_1 \wedge \alpha_3 + \frac{1 - r - s}{2}\alpha_1 \wedge \alpha_4 + \frac{1 + r + s}{2}\alpha_2 \wedge \alpha_3 + t\alpha_2 \wedge \alpha_4. \end{aligned} \tag{10}$$

Let us suppose first $B_{1\bar{2}} = 1$ and $B_{2\bar{1}} = 0$. From (10) we get directly the Lie algebra $\mathfrak{h}_7$ in this case.

On the other hand, let $J$ be the complex structure defined by $B_{1\bar{2}} = 1$ and $B_{2\bar{1}} = 2$. It follows from (10) that $d\alpha_4 = \alpha_1 \wedge \alpha_2$, $d\alpha_5 = -\alpha_2 \wedge \alpha_4$ and $d\alpha_6 = -\alpha_1 \wedge \alpha_4 + 2\alpha_2 \wedge \alpha_3$. Now, if we consider the change of basis $\beta_1 = -\alpha_2$, $\beta_2 = \alpha_1$, $\beta_3 = -2\alpha_3$ and $\beta_j = \alpha_j$ for $j = 4, 5, 6$, then we get that the Lie algebra underlying $J$ is $(0, 0, 0, 12, 14, 13+42) = \mathfrak{h}_{14}$.

From (6) we have that $d\Omega = 0$ if and only if $b_3 = c_3 = 0$ and $b_2 = a_3 - \bar{a}_3$. Moreover, $\Omega^3 \neq 0$ if and only if $|a_3|^2(a_3 - \bar{a}_3) \neq 0$. Thus, the set of compatible symplectic forms is nonempty and has dimension 5, by Lemma 3.4. This completes the proof of (i).

Now, if $B_{1\bar{2}} = 0$, $B_{2\bar{1}} = -1$ then from (10) we get that the underlying Lie algebra is $\mathfrak{h}_{10}$. Moreover, the Lie algebra $\mathfrak{h}_{11}$ is obtained when $B_{1\bar{2}} = 2$ and $B_{2\bar{1}} = 1$, which follows directly from (10) by multiplying $\alpha_3$ and $\alpha_5$ by $-2$, and changing the sign of $\alpha_6$.

If $B_{1\bar{2}} = 1+\mathrm{i}$ and $B_{2\bar{1}} = -\mathrm{i}$, then we can consider the change of basis $\beta_4 = \alpha_4+\alpha_3$, $\beta_6 = \alpha_6 - \alpha_5$, $\beta_j = \alpha_j$ for $j \neq 4, 6$, and it follows from (10) that $\beta_1, \beta_2, \beta_3$ are closed, $d\beta_4 = \beta_1 \wedge \beta_2$, $d\beta_5 = \beta_1 \wedge \beta_3$ and $d\beta_6 = \beta_2 \wedge \beta_4$, that is, the Lie algebra is $\mathfrak{h}_{12}$.

If $B_{1\bar{2}} = -1 + \mathrm{i}$ and $B_{2\bar{1}} = -\mathrm{i}$, then from Eq. (10) we get that $\mathrm{d}\alpha_5 = -\alpha_2 \wedge \alpha_4$ and $\mathrm{d}\alpha_6 = \alpha_1 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_4$. Thus, if we consider the change of basis given by $\beta_5 = \alpha_5 + \alpha_6$, $\beta_6 = -\alpha_5$, $\beta_j = \alpha_j$ for $j \neq 5, 6$, then we conclude that the underlying Lie algebra is $\mathfrak{h}_{13}$.

Now, from (6) we have that $c_3 = 0$ because $B_{12} = 1$ in any case. Also, from $(13\bar{1})_0$ in (6) we get $b_3 = 0$, because $B_{1\bar{1}} = 0$ and $A_{1\bar{1}} = 1$. It remains to consider the equation $(12\bar{1})_0$, that is, $b_2 = a_3\bar{B}_{1\bar{2}} - \bar{a}_3$. Since $b_2 + \bar{b}_2 = 0$, the coefficient $a_3$ must satisfy $a_3(\bar{B}_{1\bar{2}} - 1) + \bar{a}_3(B_{1\bar{2}} - 1) = 0$. Since $B_{1\bar{2}} = 0, 2, 1 + \mathrm{i}$ or $-1 + \mathrm{i}$, the latter equation is nontrivial.

Finally, in order to complete the proof of (ii), notice that $\Omega^3 \neq 0$ if and only if $|a_3|^2(a_3\bar{B}_{1\bar{2}} - \bar{a}_3) \neq 0$. Therefore, the set of compatible symplectic forms is nonempty and has dimension 4 by Lemma 3.4. □

**Proposition 3.10.** *There are abelian complex structures* $J$ *on* $\mathfrak{h}_{15} = (0, 0, 0, 12, 13 + 42, 14 + 23)$ *having compatible symplectic forms. Moreover*, $\dim \mathcal{S}_c(\mathfrak{h}_{15}, J) = 4$ *for any such* $J$.

**Proof.** It is proved in [5] that any abelian complex structure on $\mathfrak{h}_{15}$ having compatible $\Omega$ is given, up to complex isomorphism, by the equations:

$$\mathrm{d}\omega_1 = 0, \qquad \mathrm{d}\omega_2 = \omega_1 \wedge \bar{\omega}_1, \qquad \mathrm{d}\omega_3 = \omega_1 \wedge \bar{\omega}_2 + B_{2\bar{1}}\omega_2 \wedge \bar{\omega}_1,$$

where $|B_{2\bar{1}}| \neq 1$. Now, from (6) and (7) it follows that $\Omega$ is closed and nondegenerate if and only if $b_3 = c_3 = 0$ and $a_3 = b_2 \neq 0$, which implies that $\mathcal{S}_c(\mathfrak{h}_{15}, J)$ is four-dimensional for any such an abelian $J$. □

**Remark 3.11.** Let us remark that $\mathfrak{h}_{15}$ possesses abelian complex structures $J$ not admitting compatible $\Omega$. In fact, in [5] it is shown that any such a $J$ is, up to isomorphism, defined by: $\mathrm{d}\omega_1 = 0$, $\mathrm{d}\omega_2 = \omega_1 \wedge \bar{\omega}_1$, $\mathrm{d}\omega_3 = \omega_2 \wedge \bar{\omega}_1$.

## 4. Curvature of pseudo-Kähler metrics

Since any pseudo-Kähler metric on a nilpotent Lie algebra is Ricci flat [6], our goal in this section is to show how the curvature tensor $R$ varies when we perform a deformation of the metric.

Next we recall some basic definitions, adapted to our setting. First of all, since we are working at the level of a Lie algebra $\mathfrak{g}$, the Koszul formula for the Levi–Civita connection $\nabla$ of a metric $g$, extended to the complexification $\mathfrak{g}^{\mathbb{C}}$ of the Lie algebra, reduces to

$$2g(\nabla_X Y, T) = g([X, Y], T) - g([Y, T], X) + g([T, X], Y)$$

for $X, Y, T \in \mathfrak{g}^{\mathbb{C}}$.

Let us consider a pseudo-Kähler metric $g$ given by (8) with respect to the basis $\{\omega_1, \omega_2, \omega_3\}$ for $\mathfrak{g}^{1,0}$. We shall express $\nabla$ in terms of its dual basis $\{Z_1, Z_2, Z_3\}$ for $\mathfrak{g}_{1,0}$ and its complex conjugate. Notice that $\nabla_{\bar{Z}_k}\bar{Z}_j = \overline{\nabla_{Z_k}Z_j}$ and $\nabla_{Z_k}\bar{Z}_j = \overline{\nabla_{\bar{Z}_k}Z_j}$, because $g$ is real, therefore it suffices to compute $\nabla_{Z_k} Z_j$ and $\nabla_{\bar{Z}_k} Z_j$, for $j, k = 1, 2, 3$. Moreover, $(g, J)$ is a pseudo-Kähler structure, so $\nabla J = 0$, i.e. $\nabla_X(JY) = J(\nabla_X Y)$, for $X, Y \in \mathfrak{g}^{\mathbb{C}}$. This implies $\nabla_X Y \in \mathfrak{g}_{1,0}$, whenever $Y \in \mathfrak{g}_{1,0}$. In particular, $\nabla_X Z_j$ has type $(1, 0)$ with respect to the complex structure, for $j = 1, 2, 3$.

We shall also compute the curvature tensor $R$ of $g$, which is given by

$$R_{XYUV} = g(\nabla_{[X,Y]}U - [\nabla_X, \nabla_Y]U, V)$$

for $X, Y, U, V \in \mathfrak{g}^{\mathbb{C}}$, in terms of the basis $\{Z_j, \bar{Z}_j\}_{j=1}^3$ for $\mathfrak{g}^{\mathbb{C}}$. Notice that if $U \in \mathfrak{g}_{1,0}$ then $R_{XYUV}$ is zero for any $V \in \mathfrak{g}_{1,0}$. By the symmetries of the curvature and the fact that $R_{\bar{X}\bar{Y}\bar{U}\bar{V}} = \overline{R_{XYUV}}$, we conclude that $g$ is nonflat if and only if $R_{X\bar{Y}U\bar{V}} \neq 0$ for some $X, Y, U, V \in \mathfrak{g}_{1,0}$. Therefore, to study the flatness of a metric $g$ given by (8) it suffices to calculate $R_{Z_j\bar{Z}_k Z_l \bar{Z}_m}$.

It is clear that the existence of flat or nonflat pseudo-Kähler metrics on a Lie algebra $\mathfrak{g}$ is a property which is invariant under complex isomorphisms, in the following sense. Let $J, J'$ be two isomorphic complex structures on $\mathfrak{g}$. Then, there is a nonflat (resp. flat) pseudo-Kähler metric on $\mathfrak{g}$ compatible with $J$ if and only if there is a nonflat (resp. flat) pseudo-Kähler metric on $\mathfrak{g}$ compatible with $J'$.

From now on, we restrict our attention to abelian complex structures $J$, that is to pseudo-Kähler metrics whose underlying complex structure is abelian. In dimension 6, there are four (nonisomorphic) Lie algebras having such a pseudo-Kähler metric $g$, namely $\mathfrak{h}_5, \mathfrak{h}_8, \mathfrak{h}_9$ and $\mathfrak{h}_{15}$ (see [5] for details). We shall prove next that the flatness of $g$ only depends on its underlying complex structure.

**Theorem 4.1.** *Let $\mathfrak{g}$ be a (nonabelian) nilpotent Lie algebra of dimension 6, and $g$ a pseudo-Kähler metric on $\mathfrak{g}$ whose underlying complex structure is abelian. Then, the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_5, \mathfrak{h}_8, \mathfrak{h}_9$ or $\mathfrak{h}_{15}$. Moreover*:

(i) *Any such a metric $g$ on $\mathfrak{h}_8$ is flat.*
(ii) *Any metric $g$ on the Lie algebras $\mathfrak{h}_5$ and $\mathfrak{h}_9$ is Ricci flat but nonflat.*
(iii) *The Lie algebra $\mathfrak{h}_{15}$ has both flat and Ricci flat nonflat metrics $g$. Moreover, fixed an abelian complex structure $J$ on $\mathfrak{h}_{15}$, the corresponding pseudo-Kähler metrics are all either flat or Ricci flat nonflat.*

**Proof.** As it has been proved in Proposition 3.7, any complex structure on $\mathfrak{h}_8$ is abelian and can be expressed, up to complex isomorphism, by the equations:

$$\mathrm{d}\omega_1 = \mathrm{d}\omega_2 = 0, \qquad \mathrm{d}\omega_3 = \omega_1 \wedge \bar{\omega}_1.$$

The corresponding pseudo-Kähler metrics $g$ are given by (8) with $b_3 = c_3 = 0$ and $a_3, b_2 \neq 0$. Now direct calculations in terms of the basis $\{Z_1, Z_2, Z_3\}$, dual to the basis $\{\omega_1, \omega_2, \omega_3\}$, show that the Levi–Civita connection is given, up to complex conjugation, by

$$\nabla_{Z_1} Z_1 = \frac{a_3}{\bar{a}_3} Z_3, \qquad \nabla_{\bar{Z}_1} Z_1 = Z_3$$

with the rest vanishing. Now it is easy to check that the curvature tensor $R$ of any such a metric $g$ vanishes identically. This completes the proof of (i).

Next let us prove (ii) for $\mathfrak{h}_5$. From Proposition 3.6, we know that any two abelian complex structures on $\mathfrak{h}_5$ having a compatible pseudo-Kähler metric are isomorphic, and can be expressed by the equations:

$$d\omega_1 = d\omega_2 = 0, \qquad d\omega_3 = \omega_1 \wedge \bar{\omega}_2. \tag{11}$$

The corresponding metrics $g$ given by (8) satisfy $a_3 = c_3 = 0$ and $a_1, b_3 \neq 0$. A direct calculation in terms of the basis $\{Z_1, Z_2, Z_3\}$, dual to $\{\omega_1, \omega_2, \omega_3\}$, shows that the Levi–Civita connection is given, up to complex conjugation, by

$$\nabla_{Z_2} Z_2 = \frac{-b_3}{a_1} Z_1 - \frac{a_2 b_3}{a_1 \bar{b}_3} Z_3, \qquad \nabla_{\bar{Z}_2} Z_1 = Z_3$$

with the rest vanishing. Now it is easy to check that the only nonzero component of the curvature tensor is

$$R_{Z_2 \bar{Z}_2 Z_2 \bar{Z}_2} = -\mathrm{i}\frac{|b_3|^2}{a_1}.$$

Since $\mathrm{i}a_1$ and $|b_3|$ are nonzero real numbers for any pseudo-Kähler metric $g$, its curvature does not vanish. Let us notice that, for the Lie algebra $\mathfrak{h}_5$, if $b_2 \neq 0$ then the two-dimensional subspace generated by $Z_2$ and $\bar{Z}_2$ is nondegenerate and has nonzero sectional curvature, while all the other sectional curvatures vanish.

Finally, we prove (ii) for $\mathfrak{h}_9$, and (iii). As it has been proved in Proposition 3.8, any complex structure on $\mathfrak{h}_9$ is abelian and can be expressed, up to complex isomorphism, by the equations:

$$d\omega_1 = 0, \qquad d\omega_2 = \omega_1 \wedge \bar{\omega}_1, \qquad d\omega_3 = \omega_1 \wedge \bar{\omega}_2 + B_{2\bar{1}}\omega_2 \wedge \bar{\omega}_1, \tag{12}$$

where $|B_{2\bar{1}}| = 1$. On the other hand, from Proposition 3.10 we know that any abelian complex structure on $\mathfrak{h}_{15}$ having compatible pseudo-Kähler metrics is given, up to complex isomorphism, by the same equations, but with $|B_{2\bar{1}}| \neq 1$. In both cases, the corresponding metrics $g$ are given by (8) and satisfy $b_3 = c_3 = 0$, and $a_3 = b_2 \neq 0$.

A straightforward computation in terms of the basis $\{Z_1, Z_2, Z_3\}$ dual to $\{\omega_1, \omega_2, \omega_3\}$ shows that the Levi–Civita connection is given, up to complex conjugation, by

$$\begin{aligned} \nabla_{Z_1} Z_1 &= -\bar{B}_{2\bar{1}} Z_2 + \frac{a_2 + \bar{a}_2 \bar{B}_{2\bar{1}}}{\bar{a}_3} Z_3, \qquad \nabla_{Z_1} Z_2 = \nabla_{Z_2} Z_1 = \frac{a_3}{\bar{a}_3} Z_3, \\ \nabla_{\bar{Z}_1} Z_1 &= Z_2, \qquad \nabla_{\bar{Z}_1} Z_2 = B_{2\bar{1}} Z_3, \qquad \nabla_{\bar{Z}_2} Z_1 = Z_3 \end{aligned}$$

with the rest vanishing. Using these expressions, it is easy to check that the components of the curvature tensor $R$ are all zero, except possibly

$$R_{Z_1 \bar{Z}_1 Z_1 \bar{Z}_1} = \mathrm{i}\bar{a}_3(3 - |B_{2\bar{1}}|^2).$$

Therefore, the metric is flat if and only if $|B_{2\bar{1}}|^2 = 3$, because $a_3$ is nonzero.

Since the underlying Lie algebra is $\mathfrak{h}_9$ if and only if $|B_{2\bar{1}}| = 1$, we conclude (ii) for $\mathfrak{h}_9$.

Finally, if $|B_{2\bar{1}}| = \sqrt{3}$ then any pseudo-Kähler metric associated to the complex structure defined by (12) is flat, and if $|B_{2\bar{1}}| \neq 1, \sqrt{3}$, then the pseudo-Kähler metrics are nonflat. This proves (iii).

Again, we must notice that in both cases, for $\mathfrak{h}_9$ and $\mathfrak{h}_{15}$, if the metric is nonflat and $a_1 \neq 0$ then the two-dimensional subspace generated by $Z_1$ and $\bar{Z}_1$ is nondegenerate and it is the only with nonzero sectional curvature. □

In [6] the authors define a pseudo-Kähler metric on $\mathfrak{h}_5$ whose underlying complex structure is abelian, and which is nonflat; our result above says that this happens for any such a metric.

From Propositions 3.7 and 3.8, any complex structure on $\mathfrak{h}_8$ and $\mathfrak{h}_9$ is abelian. Therefore we have the following corollary.

**Corollary 4.2.** *Any pseudo-Kähler metric on the Lie algebra* $\mathfrak{h}_8$ *is flat, whereas any pseudo-Kähler metric on* $\mathfrak{h}_9$ *is nonflat.*

**Remark 4.3.** In dimension 4, there is (up to isomorphism) only one nonabelian nilpotent Lie algebra having complex structures. This Lie algebra, which we denote by $\mathfrak{Kt}$, is the one underlying the well-known Kodaira–Thurston manifold [9]. It is easy to prove that any complex structure $J$ on $\mathfrak{Kt}$ can be expressed by the complex equations $d\omega_1 = 0$ and $d\omega_2 = \omega_1 \wedge \bar{\omega}_1$, so such a $J$ is always abelian. Now, a direct calculation shows that $\dim \mathcal{S}_c(\mathfrak{Kt}, J) = 3$, and any pseudo-Kähler metric $g$ on $\mathfrak{Kt}$ is given by

$$g = -i(a_1\omega_1 \# \bar{\omega}_1 + a_2\omega_1 \# \bar{\omega}_2 - \bar{a}_2\omega_2 \# \bar{\omega}_1),$$

where $a_2 \in \mathbb{C} - \{0\}$ and $ia_1 \in \mathbb{R}$. It is easy to check that all these metrics are flat. Notice that the six-dimensional Lie algebra $\mathfrak{h}_8$ is a trivial extension of $\mathfrak{Kt}$.

In the table below, we show the dimension of the set of pseudo-Kähler metrics with abelian underlying complex structure on nilpotent Lie algebras, together with information on their curvature:

Pseudo-Kähler metrics with $J$ abelian in dimension $\leq 6$

| Algebra | Structure | $\dim \mathcal{S}_c$ | Ric | $R$ |
|---|---|---|---|---|
| $\mathfrak{a}^{2k}$ | $(0, \overset{2k}{\ldots}, 0)$ | $k^2$ | 0 | 0 |
| $\mathfrak{Kt}$ | (0, 0, 0, 12) | 3 | 0 | 0 |
| $\mathfrak{h}_5$ | (0, 0, 0, 0, 13 + 42, 14 + 23) | 6 | 0 | $\neq 0$ |
| $\mathfrak{h}_8 = \mathfrak{Kt} \times \mathfrak{a}^2$ | (0, 0, 0, 0, 0, 12) | 6 | 0 | 0 |
| $\mathfrak{h}_9$ | (0, 0, 0, 0, 12, 14 + 25) | 4 | 0 | $\neq 0$ |
| $\mathfrak{h}_{15}$ | (0, 0, 0, 12, 13 + 42, 14 + 23) | 4 | 0 | $0, \neq 0$ |

**Remark 4.4.** There are nonflat pseudo-Kähler metrics whose underlying complex structure is nonabelian. For example, let us consider the Lie algebra $\mathfrak{h}_2$ with the complex structure $J$ defined by Eq. (9) with $B_{2\bar{1}} = 1$. From the proof of Proposition 3.5, any pseudo-Kähler metric $g$ on $\mathfrak{h}_2$ compatible with $J$ is given by (8) with $a_3 = \bar{a}_3, b_3 = -\bar{b}_3, c_3 = 0$ and $\mu = (a_3)^2 b_2 - (b_3)^2 a_1 - a_3 b_3 (a_2 + \bar{a}_2) \neq 0$. A direct calculation shows that the components

$R_{Z_j\bar{Z}_jZ_j\bar{Z}_j}$ $(j = 1, 2)$ of the curvature tensor $R$ of $g$ are

$$R_{Z_1\bar{Z}_1Z_1\bar{Z}_1} = \frac{-\mathrm{i}(a_3)^4}{\mu}, \qquad R_{Z_2\bar{Z}_2Z_2\bar{Z}_2} = \frac{-\mathrm{i}(b_3)^4}{\mu}.$$

Since $a_3$ and $b_3$ cannot vanish simultaneously, we conclude that any pseudo-Kähler metric on $\mathfrak{h}_2$ compatible with the nilpotent (nonabelian) complex structure $J$ is nonflat.

### 4.1. Deformation of pseudo-Kähler metrics

Here we show how to construct curves of pseudo-Kähler metrics and study the variation of the curvature tensor of these metrics when we move along the curves. According to Theorem 4.1, we shall only consider the Lie algebra $\mathfrak{h}_{15}$, although similar constructions can be done on any of the remaining algebras in Theorem 3.1.

Let $\{X_1, \dots, X_6\}$ be the basis for $\mathfrak{h}_{15}$ for which the nonzero defining bracket relations are

$$[X_1, X_2] = -X_4, \qquad -[X_1, X_3] = [X_2, X_4] = X_5,$$
$$[X_1, X_4] = [X_2, X_3] = -X_6.$$

Let $f : I \to \mathbb{R}$ be a continuous function defined on some connected interval $I \subset \mathbb{R}$ such that $f(t) \neq \pm 1$, for all $t \in I$. We define the almost complex structure $J_t$ on $\mathfrak{h}_{15}$ given by

$$J_t = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{f(t)+1}{f(t)-1} \\ 0 & 0 & 0 & 0 & \dfrac{1-f(t)}{1+f(t)} & 0 \end{bmatrix}$$

for each $t \in I$.

Now, if $\{\alpha_1, \dots, \alpha_6\}$ denotes the basis for $(\mathfrak{h}_{15})^*$ dual to $\{X_1, \dots, X_6\}$, then the $(1, 0)$-forms $\omega_1 = \alpha_1 - \mathrm{i}\alpha_2$, $\omega_2 = 2\alpha_3 + 2\mathrm{i}\alpha_4$, and $\omega_3^t = 2(f(t) - 1)\alpha_5 - 2\mathrm{i}(f(t) + 1)\alpha_6$ constitute a basis for $(\mathfrak{h}_{15})_t^{1,0}$, and the complex structure equations are

$$\mathrm{d}\omega_1 = 0, \qquad \mathrm{d}\omega_2 = \omega_1 \wedge \bar{\omega}_1, \qquad \mathrm{d}\omega_3^t = f(t)\omega_1 \wedge \bar{\omega}_2 + \omega_2 \wedge \bar{\omega}_1. \tag{13}$$

This shows that $J_t$ is abelian, and so integrable, for each $t \in I$.

Since $A_{1\bar{1}} = B_{2\bar{1}} = 1$, $B_{1\bar{2}} = f(t)$ and the remaining coefficients all vanish, Proposition 3.10 and Remark 3.11 imply that $J_t$ has compatible symplectic forms if and only if $f(t) \neq 0$. Moreover, in this case conditions (6) and (7) imply $b_3 = c_3 = 0$ and

$b_2 = f(t)a_3 \neq 0$, and from (8) it follows that the pseudo-Kähler metrics are

$$\begin{aligned} g_{x,y,r,s,t} &= r\alpha_1\#\alpha_1 + r\alpha_2\#\alpha_2 + 4s\alpha_3\#\alpha_3 + 4s\alpha_4\#\alpha_4 + 4y\alpha_1\#\alpha_3 - 4x\alpha_1\#\alpha_4 \\ &\quad + 4s\frac{f(t)-1}{f(t)}\alpha_1\#\alpha_5 - 4x\alpha_2\#\alpha_3 - 4y\alpha_2\#\alpha_4 + 4s\frac{f(t)+1}{f(t)}\alpha_2\#\alpha_6, \end{aligned} \tag{14}$$

where $r = -\mathrm{i}a_1$, $x + \mathrm{i}y = a_2$ and $s = -\mathrm{i}b_2$, that is, $x, y, r, s \in \mathbb{R}$ and $s \neq 0$. Notice that $g_{x,y,r,s,t}(X, Y) = 0$ for any $X, Y$ in the centre of the Lie algebra.

On the other hand, it is easy to check that the corresponding symplectic forms are given by

$$\begin{aligned} \Omega_{x,y,r,s,t} &= -2r\alpha_1 \wedge \alpha_2 + 4x\alpha_1 \wedge \alpha_3 + 4y\alpha_1 \wedge \alpha_4 - 4s\frac{f(t)+1}{f(t)}\alpha_1 \wedge \alpha_6 \\ &\quad + 4y\alpha_2 \wedge \alpha_3 - 4x\alpha_2 \wedge \alpha_4 + 4s\frac{f(t)-1}{f(t)}\alpha_2 \wedge \alpha_5 + 8s\alpha_3 \wedge \alpha_4, \end{aligned}$$

where $x, y, r, s \in \mathbb{R}$ and $s$, $f(t)$ are nonzero.

**Proposition 4.5.** *Suppose that $f(t) \neq 0, \pm 1$ for all $t \in I$, and let $x, y, r, s : I \to \mathbb{R}$ be continuous functions such that $s(t)$ is nonzero for all $t \in I$. Then, the curvature of any metric in the curve $g_t = g_{x(t),y(t),r(t),s(t),t}$, $t \in I$, of pseudo-Kähler metrics given by (14) on the Lie algebra $\mathfrak{h}_{15}$ satisfies*:

(i) *If $|f(I)| \subset (1, \infty)$, then any metric in the curve $g_t$ is nonflat.*
(ii) *If $|f(I)| \subset (0, 1)$ then, $g_t$ is nonflat if and only if $f(t) \neq \pm(\sqrt{3}/3)$.*

*All the complex structures $J_t$ in case (i) induce the same orientation on $\mathfrak{h}_{15}$, which is opposite to the orientation induced by any complex structure in (ii).*

**Proof.** It follows directly from the proof of (iii) in Theorem 4.1. In fact, if we normalize the coefficients in (13) in order to get equations of the form (12), then the new coefficient $B_{2\bar{1}}$ is equal to $1/f(t)$, and the only eventually nonvanishing component of the curvature $R$ is equal to $s(t)(3f(t)^2 - 1)/f(t)^3$. Therefore, the metric $g_t$ is flat if and only if $f(t)^2 = 1/3$.

Finally, it is easy to check that the orientation form which corresponds to $J_t$ is given by

$$\frac{1+f(t)}{1-f(t)}\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_6. \qquad \square$$

Particular examples of curves of pseudo-Kähler metrics along of which the curvature has a special behaviour are the following:

(1) Let us consider $f(t) = (\sqrt{3}/3)\sin t$, for $t \in \mathbb{R}$. Since $f(t) \neq \pm 1$, we have an abelian complex structure $J_t$ on $\mathfrak{h}_{15}$ for all $t \in \mathbb{R}$. (Notice that $J_t$ has no compatible symplectic form if and only if $f(t) = 0$, i.e. $t = k\pi$, $k \in \mathbb{Z}$.) On the other hand, in view of (ii) in Proposition 4.5, if $t = (2k + 1)\pi/2$, $k \in \mathbb{Z}$, then $f(t) = \pm\sqrt{3}/3$ and the associated metric $g_t$ is flat. Finally, if $t \neq k\pi, (2k + 1)\pi/2$, for any $k \in \mathbb{Z}$, then the metric $g_t$ is nonflat.

(2) Consider now $f(t) = (2 + \cos t)/4$, for $t \in \mathbb{R}$. Since the image of $f$ is contained in the closed interval $[1/4, 3/4]$, in particular $f(t) \neq 0, \pm 1$, and therefore the corresponding complex structure $J_t$ has a compatible metric $g_t$, for all $t \in \mathbb{R}$. Moreover, there is an infinite number of isolated $t$'s such that $f(t) = \sqrt{3}/3 \in [1/4, 3/4]$, so we conclude from Proposition 4.5 that the flatness of the metric $g_t$ changes along the curve.

(3) The only eventually nonvanishing component of the curvature $R_t$ of $g_t$ takes the value $s(t)(3f(t)^2 - 1)/f(t)^3$. So, if $|f(I)| \subset (1, \infty)$ and we take $s(t) = f(t)^3/(3f(t)^2 - 1)$, then any metric $g_t$ is nonflat and the curvature $R_t$ does not vary along the curve. In an analogous way, if for example the function $f$ satisfies $f(t) \subset (0, \sqrt{3}/3)$, for all $t \in I$, then we get the same behaviour as before, but now the orientation induced by $g_t$ is opposite.

Finally, we notice that in (1) and (2) above the functions $x(t)$, $y(t)$, $r(t)$ and $s(t)$ can be chosen so that the resulting curves $g_t$ be periodic. In (3) we can also take suitable $f(t)$, $x(t)$, $y(t)$ and $r(t)$ in order to construct a periodic curve of pseudo-Kähler metrics.

### *4.2. Passing to the Lie group*

We finish this paper by showing how one can give explicitly complex coordinates (as a complex manifold) on a nilpotent Lie group, and then on the associated nilmanifold, starting from our knowledge of the complex structure on its Lie algebra. This is an standard procedure, so we only develop in detail two examples: $\mathfrak{h}_5$ and $\mathfrak{h}_{15}$ endowed with the complex structure given by (11) and (12), respectively.

Let us start from Eq. (11), and denote by $J$ the complex structure on the Lie algebra $\mathfrak{h}_5$ defined by these equations. We denote also by $J$ the associated left invariant complex structure on the simply connected nilpotent Lie group $G_5$ whose Lie algebra is $\mathfrak{h}_5$. In order to find a representation of the complex manifold $(G_5, J)$, we proceed as follows.

First, by integrating the structure equations we obtain

$$\omega_1 = \mathrm{d}u, \qquad \omega_2 = \mathrm{d}v, \qquad \omega_3 = \mathrm{d}w - \bar{v}\,\mathrm{d}u$$

for some (global) complex functions $u, v, w$ on $G_5$.

Next, by using that $\omega_j$ is left invariant for $j = 1, 2, 3$, one can deduce that left translation $L_{(a,b,c)}$ by an element of complex coordinates $(a, b, c)$ is given by

$$u \circ L_{(a,b,c)} = u + a, \qquad v \circ L_{(a,b,c)} = v + b, \qquad w \circ L_{(a,b,c)} = w + \bar{b}u + c.$$

Thus, the complex manifold $(G_5, J)$ can be realized as the Lie group of complex matrices:

$$(G_5, J) = \left\{ \begin{pmatrix} 1 & \bar{v} & w \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix} | u, v, w \in \mathbb{C} \right\}.$$

Now, in order to construct a compact complex nilmanifold that corresponds to the complex equations (11), it suffices to take the quotient of $G_5$ by the subgroup $\Gamma$ consisting of those matrices whose entries $\{u, v, w\}$ are Gaussian integers.

It is worthy to compare this realization with the Iwasawa manifold (whose underlying real Lie algebra is also $\mathfrak{h}_5$), which may be described as a similar quotient of the complex Heisenberg group.

We can proceed in an analogous way with the complex equations (12). Let $G_{15}$ be the simply connected nilpotent Lie group whose Lie algebra is $\mathfrak{h}_{15}$, and denote by $J$ the left invariant complex structure defined by (12) on $G_{15}$. By integrating these equations, we get

$$\omega_1 = \mathrm{d}u, \qquad \omega_2 = \mathrm{d}v - \bar{u}\,\mathrm{d}u, \qquad \omega_3 = \mathrm{d}w - B_{2\bar{1}}\bar{u}\,\mathrm{d}v + \left(\tfrac{1}{2}B_{2\bar{1}}\bar{u}^2 + u\bar{u} - \bar{v}\right)\mathrm{d}u$$

for some global complex functions $u, v, w$ on $(G_{15}, J)$. Moreover, since $\omega_j$ is left invariant for $j = 1, 2, 3$, the left translation $L_{(a,b,c)}$ by an element of complex coordinates $(a, b, c)$ is given by

$$\begin{aligned} &u \circ L_{(a,b,c)} = u + a, \qquad v \circ L_{(a,b,c)} = v + \bar{a}u + b, \\ &w \circ L_{(a,b,c)} = w + B_{2\bar{1}}\bar{a}v - \tfrac{1}{2}\bar{a}u^2 + \left(\tfrac{1}{2}B_{2\bar{1}}\bar{a}^2 - a\bar{a} + \bar{b}\right)u + c. \end{aligned} \tag{15}$$

Therefore, we can realize the complex manifold $(G_{15}, J)$ as the Lie group of complex matrices:

$$(G_{15}, J) = \left\{ \begin{pmatrix} 1 & B_{2\bar{1}}\bar{u} & -\frac{1}{2}\bar{u} & \bar{v} - u\bar{u} + \frac{1}{2}B_{2\bar{1}}\bar{u}^2 & w \\ 0 & 1 & 0 & \bar{u} & v \\ 0 & 0 & 1 & 2u & u^2 \\ 0 & 0 & 0 & 1 & u \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} | u, v, w \in \mathbb{C} \right\}.$$

If we take the quotient of $G_{15}$ by the subgroup $\Gamma$ consisting of those matrices whose entries $\{u, v, w\}$ are Gaussian integers, then we obtain a compact complex nilmanifold that corresponds to the complex equations (12).

If $B_{2\bar{1}} = 1/f(t)$ then one obtains a realization $(\Gamma\backslash G_{15}, J_t)$ of the compact complex nilmanifolds which correspond to the deformation given in Section 4.1.

Finally, notice that when $J$ is abelian, the compact complex nilmanifold has Lefschetz complex type $(1, 0)$, in the sense of [4].

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